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Stochastic Quantization of Bottomless Systems —Stationary quantities in a diffusive process—

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Abstract

By making use of the Langevin equation with a kernel, it was shown that the Feynman measure e^{-S} can be realized in a restricted sense in a diffusive stochastic process, which diverges and has no equilibrium, for bottomless systems. In this paper, the dependence on the initial conditions and the temporal behavior are analyzed for 0-dim bottomless systems. Furthermore, it is shown that it is possible to find *stationary* quantities.

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1. Introduction

It remains an unsettled question in quantum field theory how to deal with bottomless systems, i.e. systems whose Euclidean actions are unbounded from below. Important and familiar examples include Einstein gravity [1]. The fact that one cannot define the vacuum for such systems makes it impossible to quantize them in the usual ways. The standard path integral quantization procedure, for example, is not applicable, since the Feynman measure e^{-S} with a bottomless action S is not normalizable and the vacuum expectation values cannot be defined.

In the language of the stochastic quantization of Parisi and Wu [2], the difficulty with bottomless systems manifests itself as the absence of thermal equilibrium, where quantum theory is supposed to be realized for ordinary (normal) systems. In spite of this apparent drawback, however, it has been argued by several people that the potentiality inherent in yet to be studied stochastic dynamics may offer the possibility to properly deal with bottomless systems. Such attempts have already been started in Refs. [3] and [4], though on the basis of quite different ideas. An analytical expression of the probability distribution obtained for 0-dim bottomless systems [5] supports the idea expressed in Ref. [4], i.e. realization of the Feynman measure e^{-S} in a finite space-time region at finite fictitious time by means of the kernel-degree of freedom. We must note that here the stochastic process is of the diffusion type and has no equilibrium for bottomless systems, since the desired distribution e^{-S} is not normalizable and therefore no longer belongs to the spectrum of the Fokker–Planck operator H ; every eigenstate of H necessarily decays away in the large-time limit. Every quantity measured in this process is thus dependent on the initial conditions and is dependent on a fictitious time, in general. This is a severe problem if one tries to extract from this kind of treatment physically meaningful and sensible quantities, which should, of course, be independent of any of the stochastic dynamics.

The purpose of this paper is thus first to examine the dependence of the probability distribution on the initial conditions and to clarify the behavior of the correlation functions in this framework. It is observed that at large fictitious times the correlation functions diffuse quite universally, irrespectively of the initial distribution. This suggests the possibility of extracting such a quantity that does not depend on the initial conditions at large times. We then propose a method that enables us to extract it from the diffusive stochastic process.

2. Stochastic quantization with bottomless systems

Let us first briefly review the stochastic quantization of Parisi and Wu [2]. In stochastically quantizing the system with a Euclidean action $S[\phi]$, one sets up the Langevin equation

$$\frac{\partial}{\partial t}\phi(x, t) = -\frac{\delta S[\phi]}{\delta \phi(x, t)} + \eta(x, t), \quad (2.1)$$

which governs the stochastic dynamics of ϕ with respect to the fictitious time t . Here η is the Gaussian white noise characterized by the statistical properties

$$\langle \eta(x, t) \rangle = 0, \quad \langle \eta(x, t) \eta(x', t') \rangle = 2\delta^D(x - x')\delta(t - t'), \quad \text{etc.} \quad (2.2)$$

The quantum theory has been shown to be realized in the thermal equilibrium limit $t \rightarrow \infty$. One can equivalently work with the Fokker–Planck equation

$$\frac{\partial}{\partial t} P[\phi; t] = H[\phi] P[\phi; t], \quad (2.3)$$

with

$$H[\phi] = \int d^Dx \frac{\delta}{\delta\phi(x)} \left(\frac{\delta}{\delta\phi(x)} + \frac{\delta S[\phi]}{\delta\phi(x)} \right), \quad (2.4)$$

where $P[\phi; t]$ is the probability distribution of ϕ at time t . It is easy to show that the Fokker–Planck operator H is negative semi-definite for any S and that the eigenfunctional of H belonging to the highest zero eigenvalue is e^{-S} , which ensures the relaxation of P to the distribution $\propto e^{-S}$, irrespective of the choice of the initial distribution, provided that the spectrum includes the discrete zero. Quantum field theory is thus given in the thermal equilibrium limit of a hypothetical stochastic process.

This is, however, not the case with bottomless systems, since e^{-S} is not normalizable and hence does not belong to the spectrum of the Fokker–Planck operator H . Eigenvalues are negative definite, and the hypothetical stochastic process has no thermal equilibrium limit (i.e., it is a diffusion process).

3. Attempts to deal with bottomless systems

Although the naive application of stochastic quantization to bottomless systems does not work, some attempts to deal with them have been reported [3, 4]. Greensite and Halpern [3] assumed that the meaningful distribution function in a diffusive stochastic process for bottomless systems is the highest normalizable eigenstate of the Fokker–Planck operator. That is, they gave up using the Feynman measure e^{-S} to evaluate expectation values, since e^{-S} , which is not normalizable for the bottomless action S , cannot belong to the spectrum of the Fokker–Planck operator. Instead, they proposed to utilize its true ground state: the Feynman measure e^{-S} was abandoned. Since the true ground state is not a stationary state and it finally decays away at large times, they had to devise a way how to extract it from the diffusive stochastic process.

On the other hand, Tanaka et al. pursued the possibility of producing the distribution e^{-S} even for a bottomless action S by making use of the Langevin equation

$$\frac{\partial}{\partial t} \phi(x, t) = -K[\phi] \frac{\delta S[\phi]}{\delta\phi(x, t)} + \frac{\delta K[\phi]}{\delta\phi(x, t)} + K^{1/2}[\phi] \eta(x, t) \quad (3.1)$$

with a positive kernel functional K [4]. Here and in what follows, stochastic differential equations are of the Ito-type [6]. Note that the corresponding Fokker–Planck operator

is given by

$$H[\phi] = \int d^Dx \frac{\delta}{\delta\phi(x)} K[\phi] \left(\frac{\delta}{\delta\phi(x)} + \frac{\delta S[\phi]}{\delta\phi(x)} \right), \quad (3.2)$$

which has negative semi-definite eigenvalues for any positive kernel K , and that the thermal equilibrium distribution could again be given by e^{-S} *only if it is normalizable*. They expected that an appropriate choice of the positive kernel K may enable stochastic variables to be confined in a finite region and that the desired distribution e^{-S} could be reproduced there. Their numerical simulation of the Langevin equation (3.1) with a specific choice of the kernel K for simple 0-dim models actually seems to support their expectation.

In order to see the actual stochastic dynamics more clearly and in detail, let us restrict ourselves to 0-dim cases. The Langevin equation now reads

$$\dot{x} = -K(x)S'(x) + K^{1/2}(x)\eta, \quad (3.3)$$

and the corresponding Fokker–Planck equation is given by

$$\dot{P}(x, t) = \frac{\partial}{\partial x} K(x) \left(\frac{\partial}{\partial x} + S'(x) \right) P(x, t). \quad (3.4)$$

Here dots and primes denote differentiation with respect to the fictitious time t and x , respectively. It is shown in Ref. [5] that the Fokker–Planck equation (3.4) can be solved analytically with an appropriate choice of the kernel function K for any 0-dim action S which is unbounded from below for $x \rightarrow \pm\infty$. The choice of the kernel K

$$K(x) = e^{2S(x)} \quad (3.5)$$

yields the Green function for the Fokker–Planck equation (3.4) [5]

$$P(x, t; x_0) = e^{-S(x)} \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{f^2(x)}{4t}\right), \quad f(x) = \int_{x_0}^x dy e^{-S(y)}. \quad (3.6)$$

This satisfies the normalization condition $\int_{-\infty}^{\infty} dx P(x, t; x_0) = 1$ and the initial condition $P(x, 0; x_0) = \delta(x - x_0)$. The choice of the kernel (3.5) provides the Langevin equation (3.3)

$$\dot{x} = e^{2S(x)} S'(x) + e^{S(x)} \eta \quad (3.7)$$

with the desired drift force $e^{2S(x)} S'(x)$ which acts as a restoring force in bottomless regions (that is, for large $|x|$ with $xS'(x) < 0$).

One can confirm from Eq. (3.6) that the desired distribution e^{-S} is indeed realized, as Tanaka et al. expected. Consider the domain given by

$$D_t = \{x | f^2(x) < 2\gamma t\} \quad (3.8)$$

at a given time t with $\gamma \sim 1$ a real constant. Note that $2t$ is the variance of the distribution P as a function of $f(x)$, and therefore, outside this domain D_t , P is considered

to be vanishingly small owing to the Gaussian factor in Eq. (3.6). Within D_t , on the other hand, this factor is considered to be of order unity and we have the approximate distribution

$$P(x, t; x_0) \sim \frac{1}{Z_t} e^{-S(x)} \quad \text{for } x \in D_t, \quad (3.9)$$

where $Z_t \sim \int_{D_t} dx e^{-S(x)} = \int_{D_t} df = \sqrt{8\gamma t}$. The desired distribution e^{-S} is seen to be realized in D_t with the appropriate normalization factor Z_t . In other words, the stochastic variable x is considered to be distributed according to $P \sim e^{-S}/Z_t$, exclusively in D_t at time t .

4. Temporal behavior of the probability distribution

In order to confirm the above argument and for illustration, we consider a typical 0-dim bottomless system given by $S(x) = m^2 x^2/2 - \lambda x^4/4$ ($\lambda > 0$) and directly study the behavior of the probability distribution P as a function of t . Figure 1 displays its temporal behavior, starting from the initial (ideally delta-shaped) distribution $P(x, 0; x_0) = \delta(x - x_0)$, where $x_0 = 1.5a$ with $a = \sqrt{m^2/\lambda}$ being the position of a maximum of the action. Note that although the starting point of the stochastic variable $x = x_0$ is in the bottomless regions (i.e. $|x| > a$), the diffusion is well controlled (i.e., P remains in a finite region even at very large t) owing to the restoring force resulting from the kernel (3.5). Furthermore, we can see in Fig. 2, where $\tilde{S}(x, t; x_0) = -\ln[\sqrt{4\pi t}P(x, t; x_0)]$ [see Eq. (3.6)] is plotted together with S , that the distribution $P(x, t; x_0)$ is well approximated as e^{-S} within the domain where the stochastic variable is exclusively distributed. This is in accordance with the expectation discussed in the previous paragraph.

Even though the desired Feynman measure e^{-S} seems to be realized in the course of the stochastic process, the process is of the diffusion type and has no equilibrium, as is explicitly seen in the analytical expression (3.6). The distribution P decays as time increases, and expectation values over P have no thermal equilibrium limits and may depend in general on the choice of the initial conditions. Figure 3 shows the temporal behavior of $\langle x^2 \rangle_t = \int_{-\infty}^{\infty} dx x^2 P(x, t; x_0)$ for the same system as above but with different initial data. This quantity increases indefinitely and no thermal equilibrium limit exists. Note that possible errors that arise in the course of numerical evaluation of the integrations are very small, typically of order 10^{-6} .

It is, however, worth pointing out that the same figure (Fig. 3) also exhibits a universal behavior of the expectation values. After the initial transient time, $\langle x^2 \rangle_t$ turns out to approach a unique curve, albeit monotonically increasing with t , irrespectively of the choice of initial conditions. This may imply the possibility of extracting “physics” underlying bottomless systems through the diffusive stochastic process.

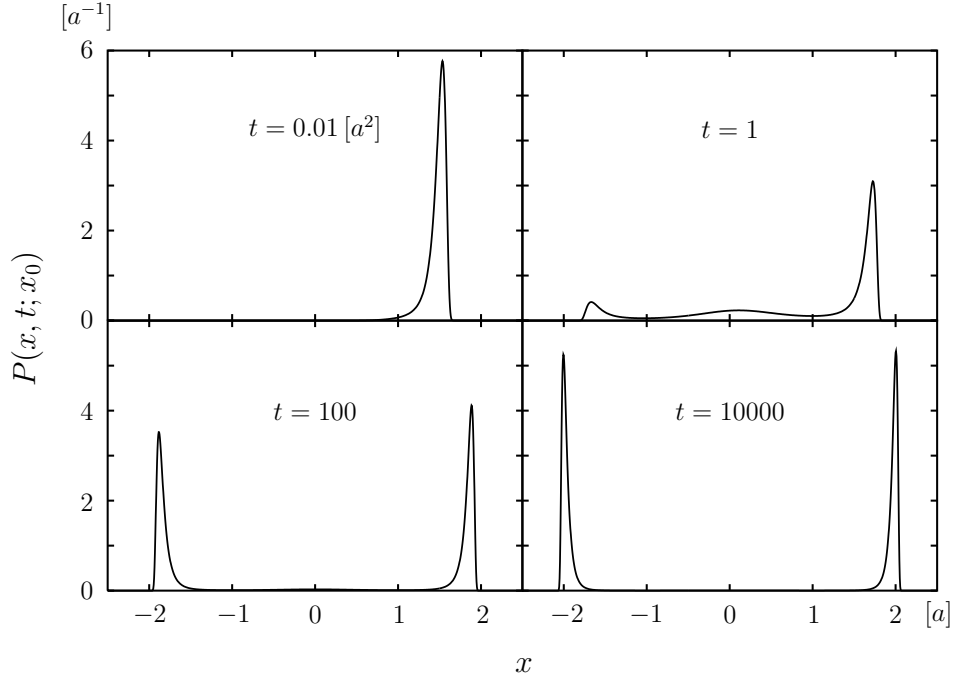


Figure 1: The distribution $P(x, t; x_0)$ for a bottomless system $S(x) = m^2 x^2/2 - \lambda x^4/4$ with $S_0 = m^4/4\lambda = 1.0$ at $t = 0.01, 1, 100$ and $10000 [a^2]$. The action has maxima S_0 at $x = \pm a = \pm\sqrt{m^2/\lambda}$. The initial distribution is a delta-shaped function located at $x_0 = 1.5a$.

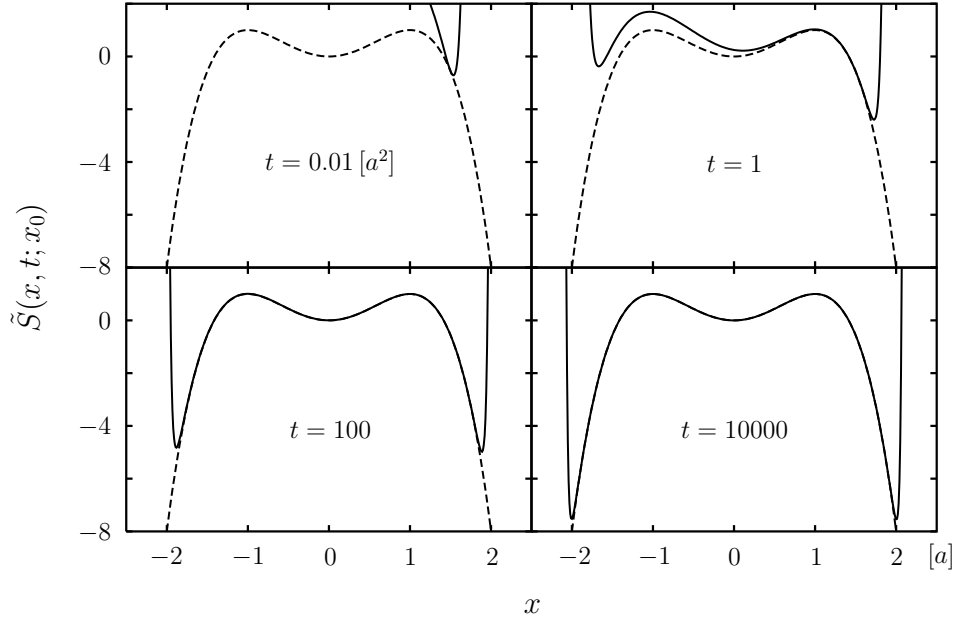


Figure 2: Temporal behavior of $\tilde{S}(x, t; x_0) = -\ln[\sqrt{4\pi t}P(x, t; x_0)]$ with the initial distribution $\delta(x - 1.5a)$. The dashed curves denote the action $S(x) = m^2 x^2/2 - \lambda x^4/4$ itself with $S_0 = m^4/4\lambda = 1.0$.

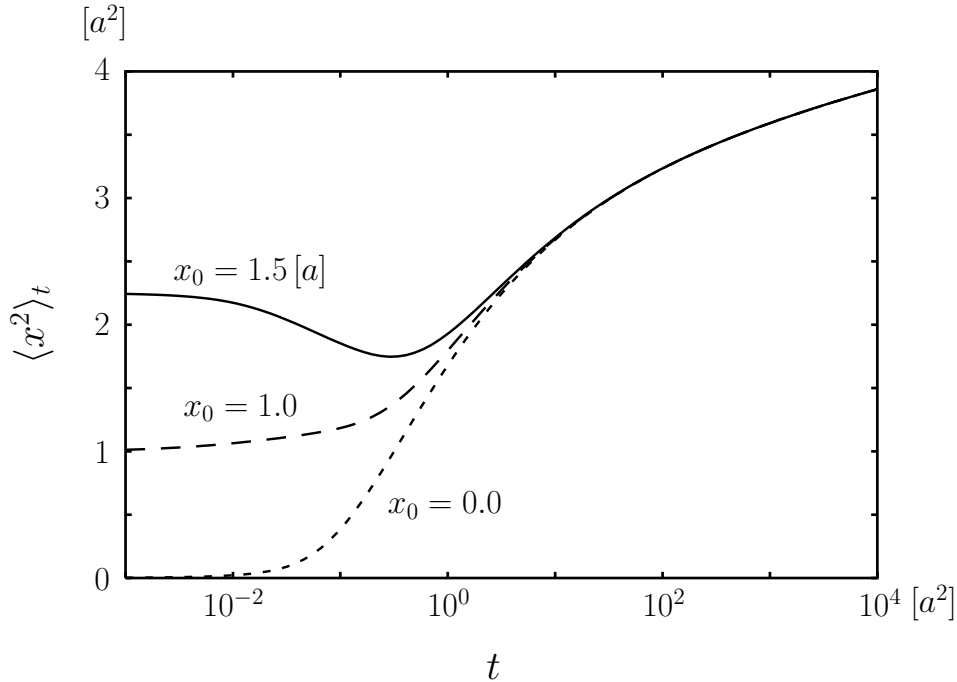


Figure 3: Temporal behavior of the expectation value $\langle x^2 \rangle_t$ for the system $S(x) = m^2 x^2/2 - \lambda x^4/4$ ($S_0 = m^4/4\lambda = 1.0$) with different initial positions $x_0 = 0.0, 1.0$ and $1.5 [a]$.

5. Stationary quantities in the diffusive process

In order to address this apparently ambitious hope, let us examine the situation more carefully. Since we do not know which quantity is to be regarded physically meaningful for bottomless systems, it seems reasonable for us to assume that a “physical quantity” within the framework of stochastic quantization is one that has a long-time limit, irrespective of the initial conditions, for any systems including bottomless ones. This is in accordance with the original spirit of stochastic quantization and may be considered a minimal requirement. [Of course, this is just one possibility, and our assumption is not meant to exclude other possibilities, such as that presented in Ref. [3].] For diffusive stochastic processes, however, whether such a quantity exists or not is a highly nontrivial question, because we naturally anticipate that every quantity should have a trivial (either vanishing or diverging) large-time limit. The idea here is to discriminate different dynamics possibly coexisting in such diffusion processes. In other words, our task is to extract appropriate “physics” underneath the dominating diffusive process caused by the unboundedness of the classical action. The following analysis is a first step along this line of thought.

Observe that the domain D_t defined by Eq. (3.8) expands monotonically as t increases. This represents one of the main features of diffusion. We attempt to estimate

this expansion quantitatively. Let $x_+(t)$ [$x_-(t)$] be the right(left) end of the domain D_t :

$$\int_{x_0}^{x_{\pm}(t)} dy e^{-S(y)} = \pm \sqrt{2\gamma t}. \quad (5.1)$$

[See Eqs. (3.8) and (3.6).] If we change the initial position x_0 by an infinitesimal amount δx_0 , it yields the following changes in $x_{\pm}(t)$:

$$\delta x_{\pm} = e^{S(x_{\pm})-S(x_0)} \delta x_0. \quad (5.2)$$

This shows that δx_{\pm} become vanishingly small as time increases, since x_+ increases and x_- decreases monotonically in time and therefore $e^{S(x_{\pm})} \rightarrow 0$. The difference between the domains D_t for different values of x_0 then disappears. This implies an approach to a unique distribution $P \propto e^{-S}$. This is the reason that we observe a unique curve for expectation values at large times in Fig. 3. It is important to note that the expansion of D_t is well controlled in this stochastic process. In fact, we can easily see, from Eq. (5.1),

$$\dot{x}_{\pm} = \pm \sqrt{\frac{\gamma}{2t}} e^{S(x_{\pm})} \sim \pm \frac{2\gamma}{Z_t} e^{S(x_{\pm})}, \quad (5.3)$$

which implies that the expansion rate diminishes exponentially at large t (or large $|x_{\pm}|$).

This observation may enable us to extract a quantity that has a long-time limit. Let us examine the time development of the expectation value of an arbitrary function $F(x)$, $\langle F(x) \rangle_t = \int_{-\infty}^{\infty} dx F(x) P(x, t; x_0)$. Using the Fokker-Planck equation (3.4), we obtain its time derivative:

$$\begin{aligned} \frac{d}{dt} \langle F(x) \rangle_t &= \int_{-\infty}^{\infty} dx F(x) \frac{\partial}{\partial x} e^{2S(x)} \left(\frac{\partial}{\partial x} + S'(x) \right) P(x, t; x_0) \\ &= \int_{-\infty}^{\infty} dx \left(\frac{\partial}{\partial x} e^{S(x)} F'(x) \right) e^{S(x)} P(x, t; x_0). \end{aligned} \quad (5.4)$$

This can be rewritten, under the validity of the approximation (3.9), as

$$\frac{d}{dt} \langle F(x) \rangle_t \sim \frac{1}{Z_t} \int_{D_t} dx \frac{\partial}{\partial x} e^{S(x)} F'(x) = \frac{1}{Z_t} e^{S(x)} F'(x) \Big|_{x_-(t)}^{x_+(t)}. \quad (5.5)$$

With the help of the equations for $x_+(t)$ and $x_-(t)$ in Eq. (5.3), it can be further reduced to

$$\frac{d}{dt} \langle F(x) \rangle_t \sim \frac{1}{2\gamma} (\dot{x}_+ F'(x_+) + \dot{x}_- F'(x_-)) = \frac{1}{2\gamma} \left(\frac{d}{dt} F(x_+) + \frac{d}{dt} F(x_-) \right). \quad (5.6)$$

One may thus expect that the quantity $\langle\langle F(x) \rangle\rangle_t$, defined by

$$\langle\langle F(x) \rangle\rangle_t = \frac{1}{2\gamma} [F(x_+(t)) + F(x_-(t))] - \langle F(x) \rangle_t, \quad (5.7)$$

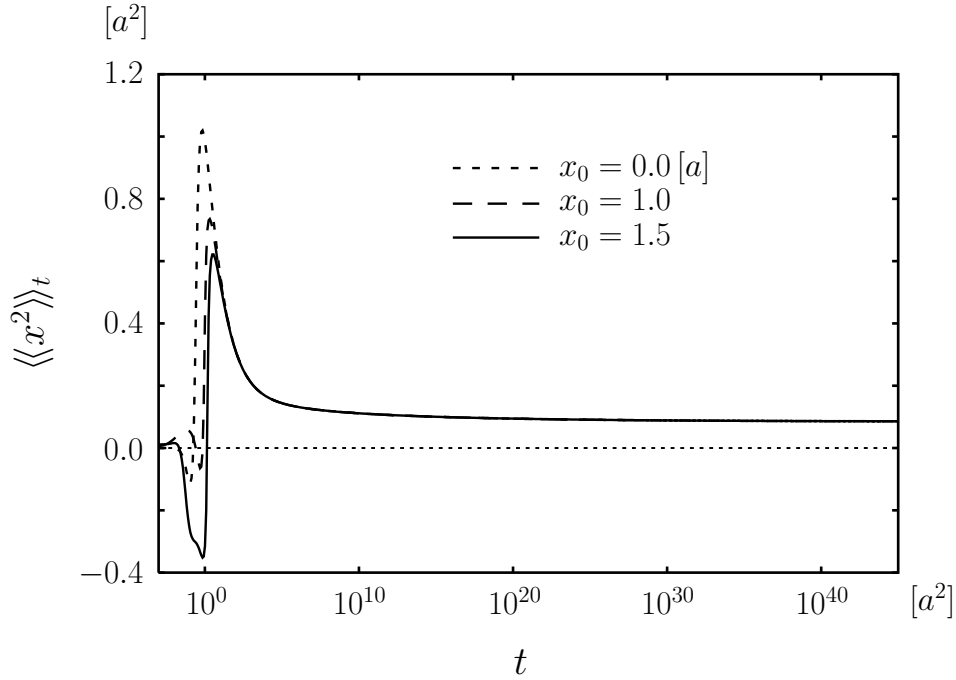


Figure 4: Temporal behavior of $\langle\langle x^2 \rangle\rangle_t$ for a bottomless system $S(x) = m^2 x^2/2 - \lambda x^4/4$ with $S_0 = m^4/4\lambda = 1.0$. Initial positions are chosen as $x_0 = 0.0, 1.0$ and 1.5 [a]. The parameter γ is chosen to be 0.995.

has a t -independent value for large t . Roughly speaking, $\langle\langle F(x) \rangle\rangle_t$ is a quantity reflecting the “quantum fluctuations” in the bottomless system, since the first two terms in the square parentheses represent “deterministic” contributions. [Remember that the dynamics for $x_{\pm}(t)$ in Eq. (5.3) are deterministic.]

The above argument has been confirmed by the numerical calculation of $\langle\langle x^2 \rangle\rangle_t$ for the system $S(x) = m^2 x^2/2 - \lambda x^4/4$ whose results are given in Fig. 4. (Here errors are typically of order 10^{-4} .) These results show that after the initial transient time, the quantity $\langle\langle x^2 \rangle\rangle_t$ approaches a certain value, which is constant over a wide range of large t , irrespectively of the choice of the initial position x_0 . Similar behavior is observed for other correlation functions.

6. Discussion

The observation of a time-independent quantity in a diffusive (or diverging) stochastic process is surprising and worthy of note. We hope that it will lead to further insight for bottomless systems. The above quantity $\langle\langle F(x) \rangle\rangle_{t \rightarrow \infty}$ is a candidate for the time-independent quantity, representing a kind of quantum effect in such systems. It can be thought, as described above, to be a measure of the quantum fluctuations for bottomless systems.

We must remember, however, that the present analysis is largely dependent on our choice of the specific kernel function $K(x)$ (3.5). In particular, the asymptotic value of $\langle\langle F(x) \rangle\rangle_t$ possibly depends on the very choice of the kernel. Even though it seems plausible that for bottomless systems meaningful quantities are rather limited, and accordingly not all conceivable kernels can provide us with them, it is very important to clarify this point, e.g. by determining the kernel dependence of $\langle\langle F(x) \rangle\rangle_{t \rightarrow \infty}$. This is an important issue to be studied, though a different choice of the kernel no longer ensures the solvability of the Fokker–Planck equation and may entail an additional numerical study in order to obtain the probability distribution P . Other points worth further investigation include (a) the relevance (if any) of $\langle\langle F(x) \rangle\rangle_t$ to observable quantities, (b) formulation on the basis of the corresponding Langevin equation (3.7), (c) comparison with other proposals [3], and (d) extension to higher dimensional systems. Work is in progress on these points.

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